I. Discriminant – Roots

Consider the quadratic equation $a x^2 + b x + c = 0$ with $(a \neq 0)$.
The number $\Delta = b^2 - 4 a c$ is called the **discriminant**.

<u>If the discriminant is positive</u>, then there exist two roots, both real numbers, computed with the formulae :

$$\alpha = \frac{-b - \sqrt{b^2 - 4 a c}}{2 a} \quad \text{and} \quad \beta = \frac{-b + \sqrt{b^2 - 4 a c}}{2 a}$$

or in short : $x = \dfrac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}$

<u>If the discriminant is zero</u>, then there exists one repeated root, computed with the formula : $x_0 = \dfrac{-b}{2 a}$

<u>If the discriminant is negative</u>, then no real number is solution, there is no root.

The diagrams below of the curve $y = a x^2 + b x + c$ shows the 3 possible cases when $a>0$:

<u>Exercise 1</u>

*Find the value of **k** for which the quadratic equation* $(5k+1)x^2 - 8 k x + 3 k = 0$ *has real roots.*

<u>Exercice 2</u>

Verify that the roots of the equation $x^2 - (a+b)x + (ab - c^2) = 0$ *are real, for all* $a, b, c \in \mathbb{R}$

<u>Vocabulary</u> discriminant

II. Sum and product of the roots of a quadratic equation

The quadratic equation $ax^2+bx+c = 0$ with $(a\neq 0)$ can be written $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Suppose roots α and β exist, the equation is $(x-\alpha)(x-\beta) = 0$.

We expand : $x^2-(\alpha+\beta)x+\alpha\beta = 0$

We now **equate** the coefficient of x and the constant term :

$$\alpha+\beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Exercise 3

If α *and* β *are the roots of the equation* $2x^2-6x+1 = 0$ *find the values of :*

$\alpha+\beta$

$\alpha\beta$

$\alpha^2+\beta^2$

$\alpha^3\beta+\alpha\beta^3$

$|\alpha-\beta|$

$\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$\alpha^3+\beta^3$

Exercise 4

Find a quadratic equation with roots $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$ *and write your answer in the form* $px^2+qx+r = 0$, *with* $p, q, r \in \mathbb{Z}$

III. Inequality proofs

This section deals with inequalities which hold for all real values of the variables.
The method is to move from the initial statement, and by using reversible steps arrive at an algebraic statement which is true.
Many times the fact that the square of any real number is greater than or equal to zero can be used.
It is important to remember that we can only square both sides of an inequality when both sides are non-negative.
Example : $2 < 5$ implies $4 < 25$ but $-3<2$ does not imply that $(-3)^2<2^2$

Exercise 5

Prove $a^2+b^2\geq 2ab$ *for all real numbers* ***a*** *and* ***b***.

Hence prove $a^2+b^2+c^2 \geq ab+bc+ca$ *for all real numbers* ***a***, ***b*** *and* ***c***.

Exercise 6

If $a^2+b^2 = 1 = c^2+d^2$, $a, b, c, d \in \mathbb{R}$ *prove that* $ab^2+cd \leq 1$

Vocabulary to equate – inequality proof