1. Sequences

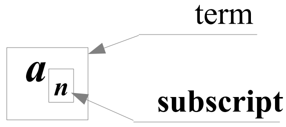
1.1. Definition

A **sequence**, or **progression**, is a succesion of numbers, called **terms**.

These terms are ordered by matching them one by one with a set of integers (or positive integers) :

the 1st term is then a_0 (*reads* « a sub **0** ») (*or* a_1), the 2nd is a_1 (*or* a_2), etc...,
and the nth term is a_{n-1} (*or* a_n), where $n \in \mathbb{N}$.

The sequence is then noted : (a_n) or : $(a_n)_{n\in I}$ where $I \subset \mathbb{N}$



There may exist an explicit rule that sets every value, or not.

If only a finite subset of $\mathbb{N}$ is matched, the sequence is **finite**. /ˈfaɪ.naɪt/

If an infinite subset of $\mathbb{N}$ (that may be $\mathbb{N}$ itself) is matched, the sequence is **infinite**. /ˈɪn.fɪ.nət/

Examples :

1.2. Definition of the term a_n

A sequence is but a function which domain is a part of $\mathbb{N}$, and codomain is included in $\mathbb{R}$.

From any function f with domain and codomain $\mathbb{R}$, a sequence can be defined by : $a_n = f(n)$,
otherwise known as the **explicit definition** of the sequence.

Conversely, any sequence can be matched to at least one function f in such a way (actually infinitely many of such functions f exist for a single sequence).

Example :

It is possible to define the **general term** a_n of a sequence in terms of some of the previous terms of the sequence itself : this is a **recurrence relation**.
Example, the famous **Fibonacci** sequence :

$a_0 = 0$
$a_1 = 1$
$a_n = a_{n-2} + a_{n-1}$ for all $n \in \mathbb{N}, n > 1$

Give the first 10 terms :

Would you think finding a function f such that $a_n = f(n)$ is an easy matter ?

1.3. Graph

The graph of a sequence is a set of points.
If a function f is known with $a_n = f(n)$, and the graph of f is drawn, then the graph of the sequence is included in that of f.

Vocabulary Fibonacci – explicit definition – finite – general term – infinite – progression – recurrence relation – sequence – subscript – term

2. Arithmetic sequences

2.1. Definition and criterion

Definition :
A sequence of numbers (a_n) is **arithmetic** if, for any positive integer n, $a_{n+1}-a_n=d$
where d is a fixed real number, called the **common difference.**

Intuitively, to go from one term to the next one, we always add the same number.

Proposition : An arithmetic sequence is defined by the first term a_0 and the recurrence relation $a_{n+1}=a_n+d$

example :

Proposition : the graph of an arithmetic sequence is included in a straight line.

2.2. Relations between terms

INDUCTION

A method of proving that each of an infinite number of statements is true, is by proving that :

i the first statement is true,

ii the truth of any one of the statements always implies the truth of the next one.

Proposition : The explicit definition of an arithmetic sequence in terms of its common difference d is :

$$a_n=a_1+(n-1)d$$

where a_1 is the first term of the sequence.

Proof (by induction) :

Proposition : For any two integers n and m :

$$a_n=a_m+(n-m)d$$

Proof (using the explicit definition) :

2.3. Sum of consecutive terms

Theorem : Let (a_n) be an arithmetic sequence.

The sum $S_n = a_1+a_2+...+a_n = \sum_{i=1}^{i=n} a_i$ of all the terms between a_1 and a_n equals : $S_n = n\frac{a_1+a_n}{2}$

Proof (using previous proposition) :

Vocabulary arithmetic – common difference – induction

2.4. Limit when *n* approaches $+\infty$

Theorem : the **limit** of an arithmetic sequence (a_n) with common difference d ...

... is equal to $+\infty$ if $d > 0$

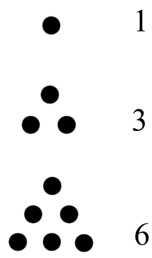
... is equal to $-\infty$ if $d < 0$

... is equal to a_1 if $d = 0$

Exercise 1

A single square is made from four matchsticks. Two squares in a row need seven matchsticks and three squares in a row need ten matchsticks. Thus is defined sequence (a_n). Determine the following :

- *a) the first term,*
- *b) the common difference,*
- *c) the explicit definition,*
- *d) how many matchsticks are in a row of 25 squares.*



Exercise 2

Drawing equilateral triangles with dots leads to the definition of triangular numbers :

*Let u_n be the difference between the **n**-th triangular number and the previous one, and $u_1 = 1$*

1. *Find the first six triangular numbers.*
2. *Give the first six terms of the sequence (u_n). What kind of a sequence could it be ?*
3. *Write the first six triangular numbers using consecutive terms of sequence (u_n)*
4. *Write the **n**-th triangular number in terms of **n**.*
5. *Find the 36th triangular number. Is this number famous for some reason?*

3. Geometric sequences

3.1. Definition and criterion

Definition :

A sequence of non zero numbers (b_n) is **geometric** if, for any positive integer n, $\frac{b_{n+1}}{b_n} = r$

where r is a fixed real number, called the **common ratio**.

A sequence of numbers (b_n) is geometric if the quotient between two consecutive terms is a constant number. Intuitively, to go from one term to the next one, we always multiply by the same number.

3.2. Relations between terms

Proposition : The explicit definition of a geometric sequence in terms of its common ratio r is :

$$b_n = b_1 \; r^{n-1}$$

where b_1 is the first term of the sequence.

Proof (by induction) :

Proposition : For any two integers n and m : $b_n = r^{n-m} \; b_m$

Proof (using the explicit definition) :

Vocabulary to approach – common ratio – geometric – limit

3.3. Sum of consecutive terms

<u>Theorem</u> : Let (b_n) be a geometric sequence.
The sum $S_n = b_1 + b_2 + \ldots + b_n = \sum_{i=1}^{i=n} b_i$ of all the terms between b_1 and b_n equals : $S_n = b_1 \dfrac{1-r^n}{1-r}$.

proof (using the explicit definition) :

<u>Exercise 3</u>
Bill and Steve decide to buy the same 2,000 € computer. They don't have the money yet, so the seller offers two types of credit. In each case, they have to pay a certain amount at the time of the actual sale, then the remainder of the 2,000 € in monthly instalments.
Bill chooses to pay 80 € first, then monthly instalments of 160 € for each of the 12 following months.
Steve chooses to pay 125 € first, and the following instalments with a monthly increase of 3 % over the next 11 months. On the twelfth month, he will pay whatever is left of the 2,000 €.

Bill's choice

*Let u_1 be the initial amount paid by Bill and u_n the total amount paid after **n** month.*
So $u_1 = 80$ and u_2 is the sum of the amounts paid at the end of the first two months.

1. *Work out u_2 and u_3 .*
2. *What kind of a sequence is (u_n) ? Explain.*
3. *Write u_n in terms of **n**.*

Steve's choice

*Let v_1 be the initial amount paid by Steve and v_n the amount paid on the **n**-th month (with **n** between 2 and 11).*
So $v_1 = 125$ and $v_2 = 129$ rounded to the closest integer.

1. *Work out v_3 , give a value rounded to the closest integer.*
2. *What kind of a sequence is (v_n) ? Explain.*
3. *What is the total amount paid by Steve at the end of the 11th month? What must he pay at the end of the 12th month?*
4. *From what time will Steve's monthly instalments be greater than Bill's?*

3.4. Limit when n approaches $+\infty$

<u>Theorem</u> : the limit of a geometric sequence (b_n) with common ratio q ...

... is equal to $+\infty$ if $q > 1$ and $b_1 > 0$
... is equal to $-\infty$ if $q > 1$ and $b_1 < 0$
... is equal to b_1 if $q = 1$
... is equal to 0 if $-1 < q < 1$ or $b_1 = 0$
... does not exist if $q \leq -1$

<u>Exercise 4</u>
A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the begining, how many bacteria will there be after 24 hours?
What becomes of this culture after a long time (the number of hours approaching infinity)? Is the assumption about the growth of the culture realistic ?

<u>Exercise 5</u>
A mine worker discovers an ore sample containing 500 mg of radioactive material. This material has a half life of 1 day. Find the amount of radioactive material at the begining of the 7th day?
What becomes of the radioactive material after a long time (the number of days approaching infinity)?
When will this sample cease to be radioactive ?