Laplacien en coordonnées polaires.

Soit f une fonction de $\mathbb{R}^2$ dans $\mathbb{R}$ de classe $\mathscr{C}^2$.
On rappelle que le Laplacien de f est la fonction :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Via le passage en coordonnées polaires, on pose :

$$f(x, y) = f(r\cos\theta, r\sin\theta) = g(r, \theta)$$

1. Calculer $\frac{\partial g}{\partial r}$, $\frac{\partial g}{\partial \theta}$, $\frac{\partial^2 g}{\partial r^2}$ et $\frac{\partial^2 g}{\partial \theta^2}$ en fonction des dérivées partielles de f (par rapport aux variables x et y).
2. Exprimer Δf en fonctions des dérivées partielles de g (par rapport aux variables r et θ).

Analyse

Un travail très classique de dérivation d'une fonction composée.

Résolution

Question 1.

On a : $g(r, \theta) = f(r\cos\theta, r\sin\theta)$.

On a donc le « schéma » : $(r, \theta) \mapsto (r\cos\theta, r\sin\theta) \mapsto f(r\cos\theta, r\sin\theta) = g(r, \theta)$.

Dans ces conditions (et avec quelques abus de notation … « classiques » ☺) :

$$\begin{aligned}\frac{\partial g}{\partial r}(r, \theta) &= \frac{\partial(r\cos\theta)}{\partial r}(r, \theta) \times \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + \frac{\partial(r\sin\theta)}{\partial r}(r, \theta) \times \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \\ &= \cos\theta \times \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + \sin\theta \times \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\end{aligned}$$

$$\boxed{\frac{\partial g}{\partial r}(r,\theta)=\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)}$$

$$\begin{aligned}\frac{\partial g}{\partial \theta}(r,\theta)&=\frac{\partial(r\cos\theta)}{\partial\theta}(r,\theta)\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+\frac{\partial(r\sin\theta)}{\partial\theta}(r,\theta)\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\\&=-r\sin\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+r\cos\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\end{aligned}$$

$$\boxed{\frac{\partial g}{\partial \theta}(r,\theta)=-r\sin\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+r\cos\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)}$$

Puis :

$$\begin{aligned}\frac{\partial^2 g}{\partial r^2}(r,\theta)&=\cos\theta\times\frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right]+\sin\theta\times\frac{\partial}{\partial r}\left[\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right]\\&=\cos\theta\times\left[\frac{\partial(r\cos\theta)}{\partial r}\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+\frac{\partial(r\sin\theta)}{\partial r}\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)\right]\\&\quad+\sin\theta\times\left[\frac{\partial(r\cos\theta)}{\partial r}\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)+\frac{\partial(r\sin\theta)}{\partial r}\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\right]\\&=\cos\theta\times\left[\cos\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)\right]\\&\quad+\sin\theta\times\left[\cos\theta\times\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\right]\end{aligned}$$

La fonction *f* étant de classe $\mathscr{C}^2$, on a : $\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)=\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)$. Il vient alors :

$$\begin{aligned}\frac{\partial^2 g}{\partial r^2}(r,\theta)&=\cos\theta\times\left[\cos\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)\right]\\&\quad+\sin\theta\times\left[\cos\theta\times\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\right]\\&=\cos^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)\\&\quad+\sin^2\theta\times\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\end{aligned}$$

$$\boxed{\frac{\partial^2 g}{\partial r^2}(r,\theta)=\cos^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta) \\ +\sin^2\theta\times\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)}$$

On a ensuite :

$$\begin{aligned}
\frac{\partial^2 g}{\partial \theta^2}(r,\theta) &= -r\frac{\partial}{\partial\theta}\left[\sin\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right]+r\frac{\partial}{\partial\theta}\left[\cos\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right]\\
&= -r\left\{\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+\sin\theta\times\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right]\right\}\\
&\quad +r\left\{-\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)+\cos\theta\times\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right]\right\}\\
&= -r\left\{\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right.\\
&\quad \left.+\sin\theta\times\left[\frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+\frac{\partial(r\sin\theta)}{\partial\theta}\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)\right]\right\}\\
&\quad +r\left\{-\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right.\\
&\quad \left.+\cos\theta\times\left[\frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)+\frac{\partial(r\sin\theta)}{\partial\theta}\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\right]\right\}\\
&= -r\left\{\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right.\\
&\quad \left.+\sin\theta\times\left[-r\sin\theta\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)+r\cos\theta\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta,r\sin\theta)\right]\right\}\\
&\quad +r\left\{-\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right.\\
&\quad \left.+\cos\theta\times\left[-r\sin\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)+r\cos\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)\right]\right\}\\
&= -r\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)-r\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\\
&\quad +r^2\sin^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta)-2r^2\sin\theta\cos\theta\times\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta)\\
&\quad +r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta)
\end{aligned}$$

$$\frac{\partial^2 g}{\partial\theta^2}(r,\theta) = -r\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) - r\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) + r^2\sin^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) - 2r^2\sin\theta\cos\theta\times\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta) + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)$$

Question 2.

Notre objectif consiste maintenant à obtenir la somme $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$.

On a facilement :

$$\begin{aligned}
r^2\times\frac{\partial^2 g}{\partial r^2}(r,\theta)+\frac{\partial^2 g}{\partial\theta^2}(r,\theta) &= r^2\times\Big[\cos^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)+2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta) \\
&\quad +\sin^2\theta\times\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\Big] \\
&\quad -r\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)-r\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \\
&\quad + r^2\sin^2\theta\times\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)-2r^2\sin\theta\cos\theta\times\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta) \\
&\quad +r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta) \\
&= r^2\times\Big[\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)+\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\Big] \\
&\quad -r\Big[\cos\theta\times\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)+\sin\theta\times\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\Big] \\
&= r^2\times\Delta f\,(r\cos\theta, r\sin\theta)-r\times\frac{\partial g}{\partial\theta}(r,\theta)
\end{aligned}$$

D'où : $\frac{\partial^2 g}{\partial r^2}(r,\theta)+\frac{1}{r^2}\times\frac{\partial^2 g}{\partial\theta^2}(r,\theta)=\Delta f\,(r\cos\theta, r\sin\theta)-\frac{1}{r}\times\frac{\partial g}{\partial\theta}(r,\theta)$.

Soit : $\Delta f\,(r\cos\theta, r\sin\theta)=\frac{\partial^2 g}{\partial r^2}(r,\theta)+\frac{1}{r^2}\times\frac{\partial^2 g}{\partial\theta^2}(r,\theta)+\frac{1}{r}\times\frac{\partial g}{\partial\theta}(r,\theta)$.

On écrit, classiquement :

$$\boxed{\Delta f=\frac{\partial^2 g}{\partial r^2}+\frac{1}{r^2}\times\frac{\partial^2 g}{\partial\theta^2}+\frac{1}{r}\times\frac{\partial g}{\partial\theta}}$$