Soit $z = x + iy$ un complexe.

Exprimer la partie réelle et la partie imaginaire de chacun des complexes suivants en fonction de la partie réelle et de la partie imaginaire de z.

$$z_1 = (1+i)z \qquad z_2 = (z+i)^2$$
$$z_3 = (z+i)(\bar{z}-i) \qquad z_4 = \frac{1}{z+i}$$

Analyse

Des manipulations assez classiques de formes algébriques…

Résolution

$$\begin{aligned} z_1 &= (1+i)z \\ &= (1+i)(x+iy) \\ &= x+iy+ix-y \\ &= (x-y)+i(x+y) \end{aligned}$$

D'où : $\text{Re}(z_1) = x - y = \text{Re}(z) - \text{Im}(z)$ et $\text{Im}(z_1) = x + y = \text{Re}(z) + \text{Im}(z)$.

$$\begin{aligned} z_2 &= (z+i)^2 \\ &= (x+iy+i)^2 \\ &= (x+i(1+y))^2 \\ &= x^2 + 2ix(1+y) - (1+y)^2 \\ &= x^2 - (1+y)^2 + 2ix(1+y) \end{aligned}$$

D'où : $\text{Re}(z_2) = x^2 - (1+y)^2 = (\text{Re}(z))^2 - (1+\text{Im}(z))^2$ et
$\text{Im}(z_2) = 2x(1+y) = 2\times\text{Re}(z)\times(1+\text{Im}(z))$.

Remarquons que $\overline{z+i} = \bar{z} + \bar{i} = \bar{z} - i$. Ainsi : $z_3 = (z+i)(\bar{z}-i) = (z+i)\overline{(z+i)} = |z+i|^2$.

Comme $z + i = x + i(1+y)$, il vient immédiatement : $z_3 = |z+i|^2 = x^2 + (1+y)^2$.

D'où : $\operatorname{Re}(z_3) = x^2 + (1+y)^2 = (\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2$ et $\operatorname{Im}(z_3) = 0$.

$$\begin{aligned} z_4 &= \frac{1}{z+i} \\ &= \frac{\overline{z}-i}{|z+i|^2} \\ &= \frac{x-i(1+y)}{x^2+(1+y)^2} \\ &= \frac{x}{x^2+(1+y)^2} + i\frac{-(1+y)}{x^2+(1+y)^2} \end{aligned}$$

D'où : $\operatorname{Re}(z_4) = \dfrac{x}{x^2+(1+y)^2} = \dfrac{\operatorname{Re}(z)}{(\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2}$ et

$\operatorname{Im}(z_4) = \dfrac{-(1+y)}{x^2+(1+y)^2} = \dfrac{-(1+\operatorname{Im}(z))}{(\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2}$.

Résultat final

$$\operatorname{Re}(z_1) = \operatorname{Re}((1+i)z) = \operatorname{Re}(z) - \operatorname{Im}(z)$$
$$\operatorname{Im}(z_1) = \operatorname{Im}((1+i)z) = \operatorname{Re}(z) + \operatorname{Im}(z)$$

$$\operatorname{Re}(z_2) = \operatorname{Re}\left((z+i)^2\right) = (\operatorname{Re}(z))^2 - (1+\operatorname{Im}(z))^2 \text{ et}$$
$$\operatorname{Im}(z_2) = \operatorname{Im}\left((z+i)^2\right) = 2\times\operatorname{Re}(z)\times(1+\operatorname{Im}(z))$$

$$\operatorname{Re}(z_3) = \operatorname{Re}((z+i)(\overline{z}-i)) = (\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2$$
$$\operatorname{Im}(z_3) = \operatorname{Im}((z+i)(\overline{z}-i)) = 0$$

$$\operatorname{Re}(z_4) = \operatorname{Re}\left(\frac{1}{z+i}\right) = \frac{\operatorname{Re}(z)}{(\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2}$$
$$\operatorname{Im}(z_4) = \operatorname{Im}\left(\frac{1}{z+i}\right) = \frac{-(1+\operatorname{Im}(z))}{(\operatorname{Re}(z))^2 + (1+\operatorname{Im}(z))^2}$$