Ptolemy's theorem

1. **Who was Ptolemy ? How can his theorem be worded (*i.e.* *w*ritten in plain English) , starting with "In a cyclic quadrilateral, the product of. . ." ?**

Ptolemy, according to the text, was a Greek astronomer and mathematician.
Further information : Claudius Ptolemy was born in AD 90 and died around AD 168. His most famous works are books « Almagest » and « Geographia », and the first known World maps.

We could write Ptolemy's theorem as :
« In a cyclic /ˈsaɪ.klɪ.k/ quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. »

2. **What is the meaning of notations ∠ , △ , and ~ ?**

∠ means angle , Δ means triangle.
~ is the relation « is similar to »

3. **Explain each step of the geometric proof, especially justifying :**

the equality of angles in 1 ;

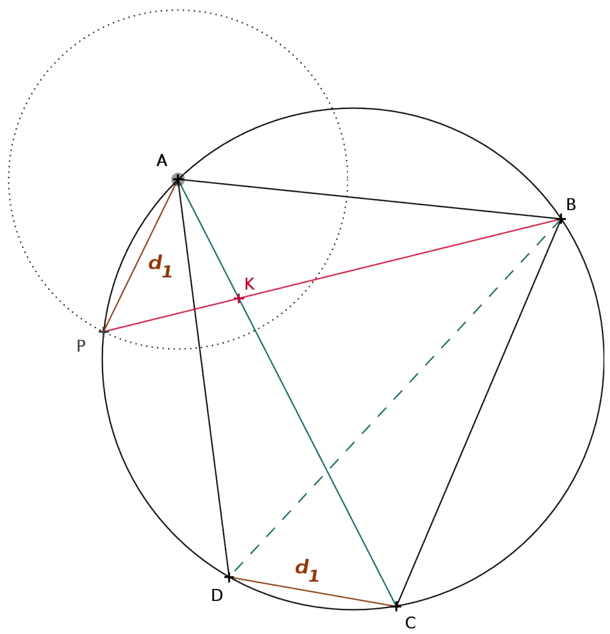
Angles $\widehat{BAC}$ and $\widehat{BDC}$ are inscribed in the circle, and are subtended by the same arc $\overset{\frown}{BC}$, then they are equal.
The same applies to $\widehat{ADB}$ and $\widehat{ACB}$, with respect to arc $\overset{\frown}{AB}$.

the construction of K in 2 ;

Let P be the intersection of line (BK) and the circle, then $\widehat{ABP} = \widehat{CBD}$

We need P on the circle such that chords [AP] and [DC] are the same length, which is obtained by drawing a circle with centre A and radius $DC = d_1$: it intersects the circle circumscribing the quadrilateral at point P.

We then draw line segment [BP], it intersects diagonal [AC] at K.



the similarity of triangles in 4 ;

K was built such that $\widehat{ABK} = \widehat{CBD}$
Arc $\overset{\frown}{BC}$ subtends both inscribed angles $\widehat{CDB}$ and $\widehat{CAB}$, the latter being $\widehat{KAB}$ as well.
Finally triangles ABK and DBC have two angles sharing the same measure, so go the third angles.

We then consider arcs $\overset{\frown}{AD}$ and $\overset{\frown}{PC}$, they are the same length, they subtend inscribed angles $\widehat{PBC}$ and $\widehat{ABD}$, which are then the same measure.
Triangles ABD and KBC have two pairs of angles sharing the same measure, they are similar.

the final calculations in 5.

Corresponding sides are : AK and DC ; KB and CB ; BA and BD.
The triangles are similar one to the other, then the corresponding sides are in the same ratio :

$$\frac{AK}{CD} = \frac{KB}{CB} = \frac{AB}{BD}$$

working $\frac{AK}{CD} = \frac{AB}{BD} \Leftrightarrow \frac{AK}{AB} = \frac{CD}{BD}$.

idem from similar triangles ABD and KBC : $\frac{CK}{BC} = \frac{DA}{BD}$

The remaining calculations are straightforward.

4. Why can you apply Ptolemy's theorem to a square ? To a rectangle ?
What do you find as a result in each case ?

A rectangle is a cyclic quadrilateral, its circumcircle is with centre the intersection of the diagonals, and with diameter the diagonal.
The same applies to the square.

Let a be the side of the square, d the diagonal, the theorem states : $d^2 = a \times a + a \times a$
which we expand and factorise as : $d^2 = 2a^2$
and we find the diagonal of a square in terms of its side : $d = a\sqrt{2}$

Let a and b be the length and breadth of a rectangle, let d be its diagonal,
the theorem states : $d^2 = a \times a + b \times b$
which we expand and factorise as : $d^2 = a^2 + b^2$ (that is Pythagoras' theorem)
and we find the diagonal of a rectangle in terms of its sides : $d = \sqrt{a^2 + b^2}$

5. Let ABCDE be a regular pentagon, which side and chord are named *a* and *b* respectively.
Find the relationship between *a* and *b*. (The chords of the pentagon are AC, BD, CE, etc.)

The sides are AB = BC = CD = DE = EA = a
The chords are AC = AD = BD = BE = CE = b

Since ABCDE is regular, it is cyclic,
hence ABCE being a cyclic quadrilateral.

The diagonals of the quadrilateral are two chords of the pentagon, with length b.
One pair of opposite sides have lengths a, the other pair have lengths a and b.

Ptolemy's theorem states :

$$b^2 = a^2 + ab$$

or, completing the square :

$$2b^2 + ab = a^2 + 2ab + b^2$$

$$\Leftrightarrow \quad b(a+2b) = (a+b)^2 \quad \ldots$$