1. Outcome - Principles of counting

The result of an operation is called an **outcome**.

Example : If we **toss** a coin, there are two possibilities for its landing, **heads** up or **tails** up : there are two possible outcomes for tossing a coin.

Fundamental principle of counting – I
Suppose an operation has ***m*** possible outcomes and that a second operation has ***n*** outcomes. The number of possible outcomes when performing the first operation followed by the second is $m \times n$

Note : **assuming the outcome of the first operation does not affect the second operation**.

Example : We cast a six-sided dice and a coin at the same time. There are 6 outcomes for the dice, and 2 for the coin. Therefore, there are $6 \times 2 = 12$ different **ordered pairs** (dice outcome ; coin outcome).

Fundamental principle of counting – II
Suppose an operation has ***m*** possible outcomes and that a second operation has ***n*** outcomes. Suppose we define a new operation that is performing one of the first two operations, selected at random. The number of possible outcomes when performing that new operation is $m+n$

***Note* : assuming the first two operations do not overlap** : in other words, both basic operations can be performed independantly.

Example : We throw but one of the objects. The outcome is one among : {1 ; 2 ; 3 ; 4 ; 5 ; 6 ; heads ; tails }

2. Permutations

A **permutation** is an arrangement, in a definite order, of a number of objects chosen from a set.

A different order in selecting the same objects makes a different permutation.

The French use the notation A_n^r , and the English ${}_nP_r$ or ${}^nP_r$ for counting the permutations of ***r*** objects selected from ***n*** distinguishable objects.

Example : how many 2-letter permutations within the set of 3 letters $\{\Phi, \Psi, \Lambda\}$?
There are 6 of them : $\Phi\Psi$, $\Psi\Phi$, $\Psi\Lambda$, $\Lambda\Psi$, $\Lambda\Phi$, $\Phi\Lambda$
The first letter can be written down in 3 ways, the second letter can be written down in 2 ways. Thus the two operations can be performed in $3 \times 2 = 6$ ways

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Exercises

1. *Two boys and three girls are to be seated in a row on a bench*
   a) *How many different ways can they be arranged ?*
   b) *How many different ways can they be arranged if they must sit :*
      (i) Girl – Boy – Girl – Boy – Girl *(ii) Boy – Girl – Girl – Girl - Boy*
2. *A permutation lock has four rings which can be rotated about an axle, and which have 10 digits each.*
   a) *How many different codes can be generated ?*
   b) *If no digit can be repeated and 0 can never be first, fond the maximum of such locks that could be manufactured if no two locks have the same code and the lock will open only when a certain code of 4 digits is in line.*

Vocabulary to cast – combinatorics – heads – ordered pair – outcome – permutation – tails – to toss

3. Factorials

<u>Definition</u> :
The product of all the positive whole numbers from n down to **1** is called **factorial** n , and is denoted by $n!$

Thus : $n! = n(n-1)(n-2)...\times 3\times 2\times 1$

Example : evaluate $10! = 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1$

$10! = 10\times 9\times 8\times 7\times 6\times 5\times 4\times 6 = 10\times 9\times 8\times 7\times 6\times 5\times 24 = 10\times 9\times 8\times 7\times 6\times 120 = 10\times 9\times 8\times 7\times 720$

$10! = 10\times 9\times 8\times 5{,}040 = 10\times 9\times 40{,}320 = 10\times 362{,}880 = 3{,}628{,}800$

The values of factorial increase in size at a very fast rate. Guess how many figures in $100!$?

Exercise

3. *how many 3-letter permutations within the set of 3 letters* $\{\Phi, \Psi, \Lambda\}$ *?*

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4. Combinations - The $\binom{n}{r}$ notation

A **combination** is a selection of a number of objects in any order.

A different order in selecting the same objects makes no difference for the resulting combination.

<u>Example</u> : how many 2-letter combinations within the set of 3 letters $\{\Phi, \Psi, \Lambda\}$?
There are 3 of them : $\Phi\Psi$, $\Psi\Lambda$, $\Lambda\Phi$. If you start counting permutations, you then divide by the number of permutations associated to the same combination : for instance $\Phi\Psi$ and $\Psi\Phi$ are one combination.

The notation $\binom{n}{r}$ (reads « n c r » or « n **choose** r ») or C_n^r (French way), or ${}_nC_r$ or ${}^nC_r$ (English way), is used for giving the number of ways of choosing r objects from n different objects. It is the number of all combinations of r objects chosen from n objects. In other words, it is the number of different subsets of size r amongst a set of size n.
Computing this number leads to the formulas :

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{that is the same as :} \qquad \binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)...\times 3\times 2\times 1}$$

<u>Note</u> : We know n choose n equals 1, and $0!$ is set to 1 in order to be able to compute this.

Exercises

4. *Solve* (i) $\binom{n}{2} = 10$ (ii) $\binom{n+1}{2} = 21$ (iii) $\binom{n+1}{2} = \binom{n}{1} + 4\binom{7}{1}$
5. *A set of 12 pupils exchange handshakes once with each other. Calculate the number of handshakes among the 12 pupils.*
6. *A class of 20 students wins a prize. Two members of the class are chosen to receive the prize. How many different pairs of students can be chosen ?*
7. *A fifth-year student has to choose four subjects from the following list : Accounting, Biology, Chemistry, Physics, French, Applied Maths and Classical studies.*
   a) *How many different choices are possible ?*
   b) *How many choices include French ? How many do not ?*
   c) *How many choices include Accounting and Biology ?*
   d) *How many choices include Applied Maths but not Chemistry ?*
8. *Ten points are taken on the circumference of a circle. Chords are drawn from those points.*
   a) *Draw a figure.*
   b) *Calculate the number of such chords that can be drawn.*
   c) *With these points as vertices, how many triangles can be drawn ?*
   d) *How many right-angled triangles can be drawn, provided the points are regularly plotted?*

<u>Vocabulary</u> n choose r – combination – factorial