I. Random variable

A **random variable** is a variable taking real numbers as values, according to the outcome of a random experiment.

In case of a discrete random variable, the mean of the random variable values weighted by their probability exists, it is the **expectation** (or **expected value**).

$$E(X) = \frac{\sum \left(P\left(X=x_i\right)\right) x_i}{\sum P\left(X=x_i\right)} \text{ , since } \sum P\left(X=x_i\right) = 1 \text{ the final result is } E(X) = \sum \left(P\left(X=x_i\right)\right) x_i$$

Note : In case of a continuous random variable, the expectation may not exist, and is computed through an integral.

The expectation can be interpreted as the long-run average of the results of many independent repetitions of an experiment. The expected value may be unlikely or even impossible (such as having 2.5 children), just like the sample mean.

Example : We pick a card from a 32 deck, the random variable ***X*** has value 0.5 when the card is hearts, and value 2 otherwise : $X \in \{0.5 ; 2\}$

Let $E(X)$ be the expectation of ***X***. We compute $E(X) = \frac{8}{32} \times 0.5 + \frac{24}{32} \times 2 = \frac{4+48}{32} = 1.625$

As we can see, the expectation is not a value from the range of the random variable ***X***.

II. Bernoulli trial

A **Bernoulli trial** is an experiment with two outcomes only, usually referred to as **Success** and **Failure**.

Let p be the probability of success in a Bernoulli trial. Then the probability of failure q is given by :

$$q = 1 - p$$

Let ***X*** be the random variable mapping 1 to success and 0 to failure.

Example : We cast a 6 sided-die, we call success the number 5, and everything else is failure.

$P(X=1) = \frac{1}{6}$ and $P(X=0) = \frac{5}{6}$

Exercise

1. *Prove that the expectation of the random variable associated to a Bernoulli trial with probability of success **p** is* $E(X) = p$

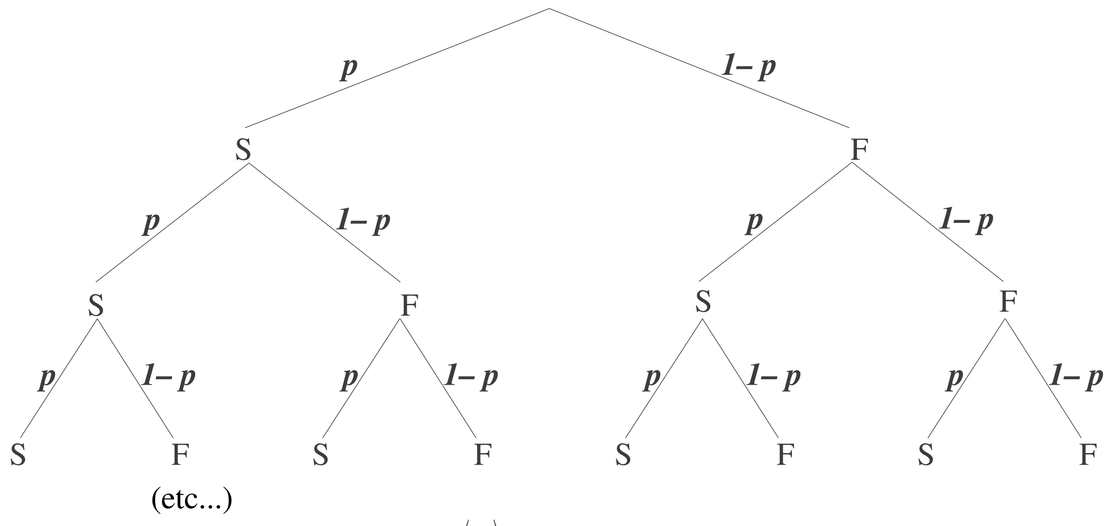
III. Binomial distribution

A **binomial experiment** consists of a fixed number n of identical and independant Bernoulli trials with p as the probability of success.

The random variable ***X*** equal to the number of success within the n trials has values in the subset of integers $\{0, 1, \ldots, n\}$

Vocabulary Bernoulli trial – binomial experiment – expectation – expected value – failure – random variable – success

A tree is a convenient diagram to represent a binomial experiment.



The number of outcomes with ***k*** successes is $\binom{n}{k}$

The probability of getting one particular outcome with ***k*** successes is $P_{k,0} = p^k (1-p)^{n-k}$

The probability of getting ***k*** successes is then $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

The **binomial distribution** is the discrete probability distribution of the number of successes in a binomial experiment. The binomial distribution is noted $\mathcal{B}(n, p)$.
The expected value of the random variable in the binomial distribution $\mathcal{B}(n, p)$ is $E(X) = np$.

Exercises

2. *A coin has been engineered, and shows heads one time in four, and heads is called success. The same coin is tossed 3 times.*
   (a) *Draw a tree matching the binomial distribution* $\mathcal{B}$ *(**3**, **0.25**)*
   (b) *Draw a table with the probabilities of the number of successes*
   (c) *Check the expectation of* $\mathcal{B}$ *(**3**, **0.25**).*

3. *A multiple choice exam is given, listing 20 questions coming each with 4 choices, exactly one of the four being the right answer.*
   Prove that a student who answers randomly (like he or she did not prepare soundly) expects a mark 5 over 20.

IV. Negative Binomial distribution

the **negative binomial distribution** is a discrete probability distribution of the number of successes in a binomial experiment before a specified number of failures (denoted ***r***) occur.
For example, if one throws a die repeatedly until the third time “1” appears, then the probability distribution of the number of non-“1”s that had appeared will be negative binomial.

Exercises

4. *A fair coin is tossed as many times as necessary. Let X be the random variable counting the number of Heads before the first Tails occurs.*
   (a) *Start a table figuring this negative binomial distribution.*
   (b) *Start another table if we want Y to count the number of Heads before the second Tails occurs.*

Vocabulary binomial distribution – negative binomial distribution